# Confidence Intervals for Proportions

## Lecture 29
## Sections 9.4

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# Outline

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# 95% Confidence Intervals

- Given the sample size $n$, the sampling distribution of $\hat{p}$ is normal with mean $p$ and standard deviation

$$\sqrt{\frac{p(1-p)}{n}}.$$

- Therefore, 95% of the time, $\hat{p}$ will lie within $1.96\sqrt{\frac{p(1-p)}{n}}$ of $p$.

# Outline

# The Target Analogy

- Consider an analogy of a shooter shooting at a target.
- Suppose a shooter hits within 5 rings (5 inches) of the bull's eye 95% of the time.
- Then each individual shot has a 95% chance of hitting within 5 inches.

# The Target Analogy

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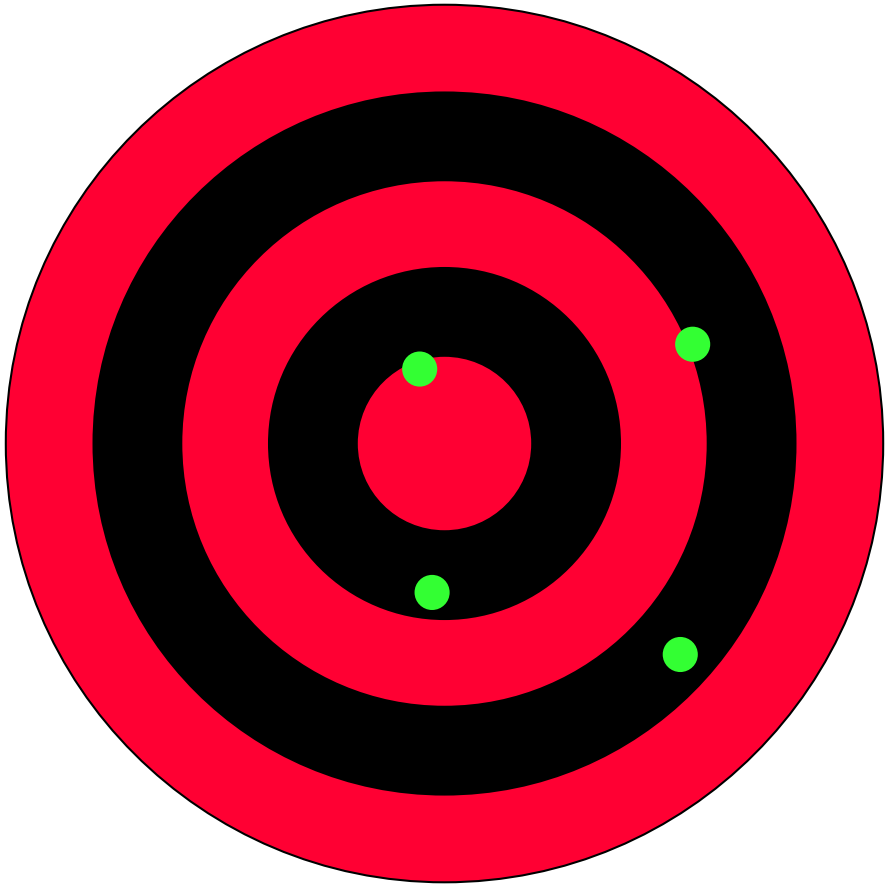

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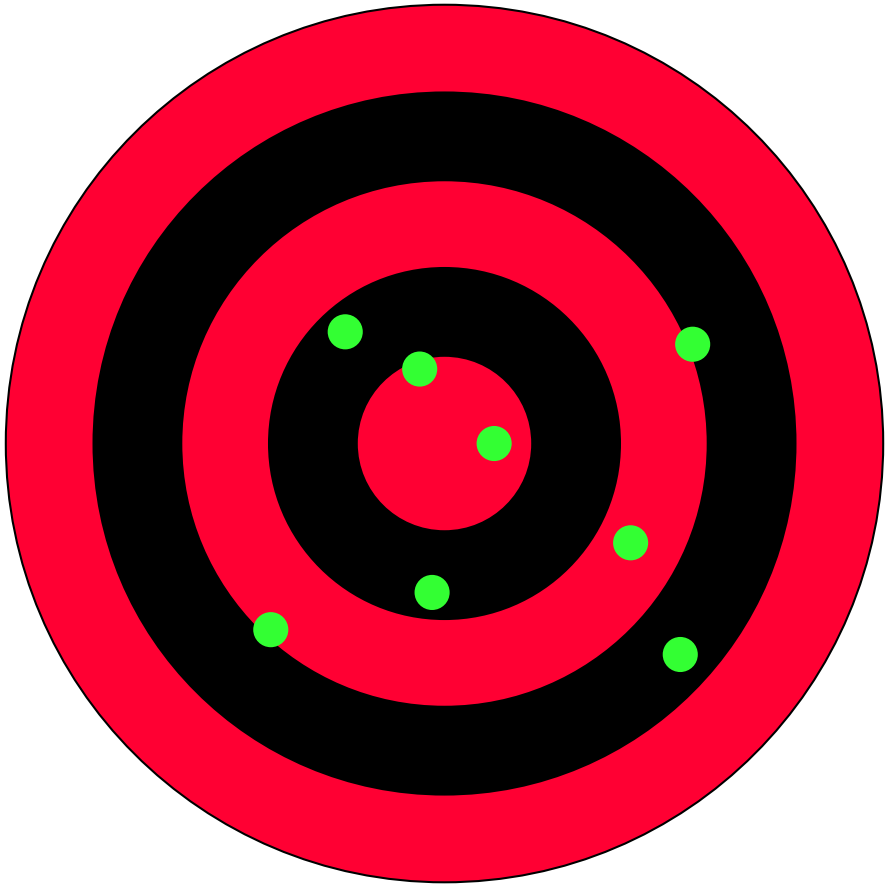

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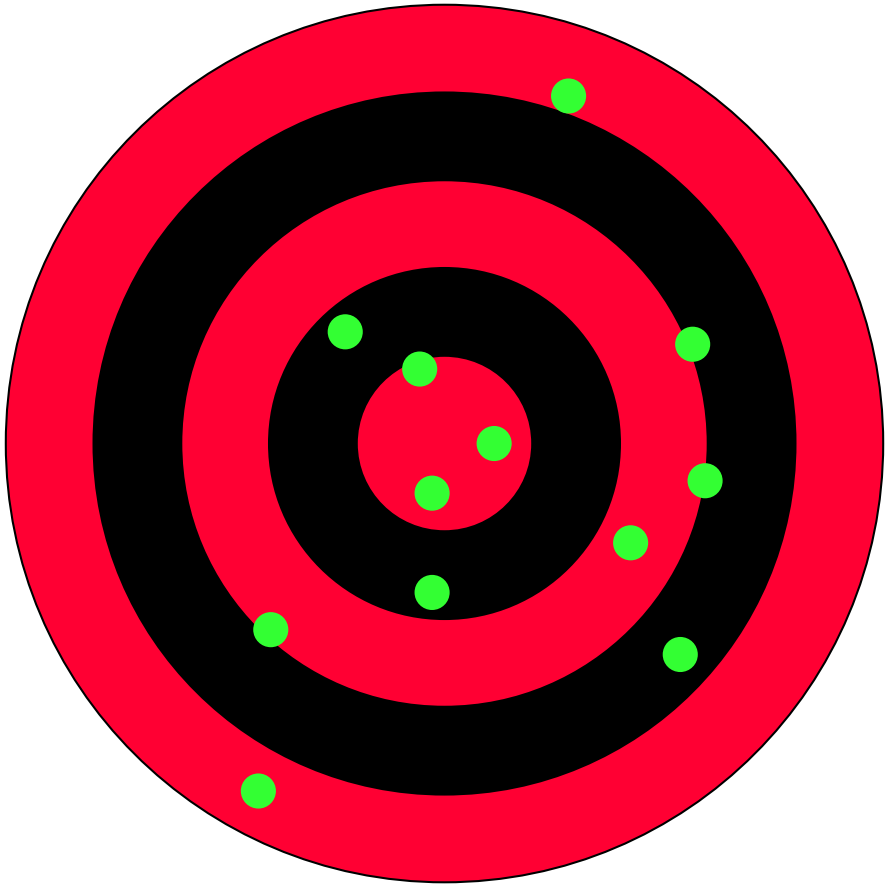

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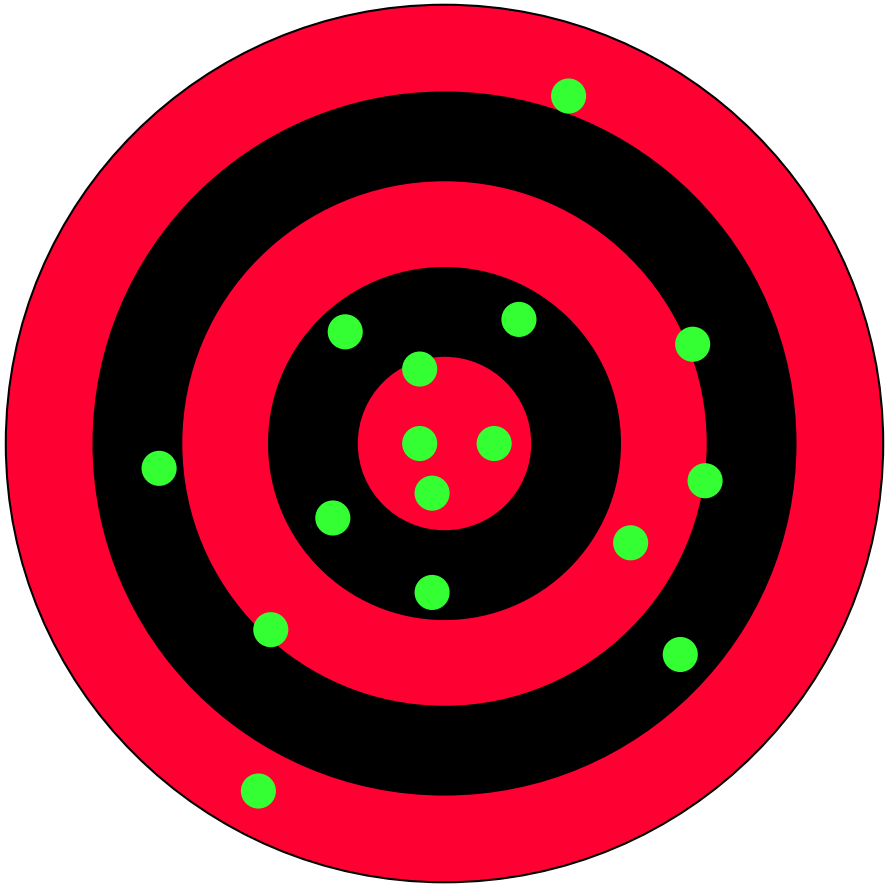

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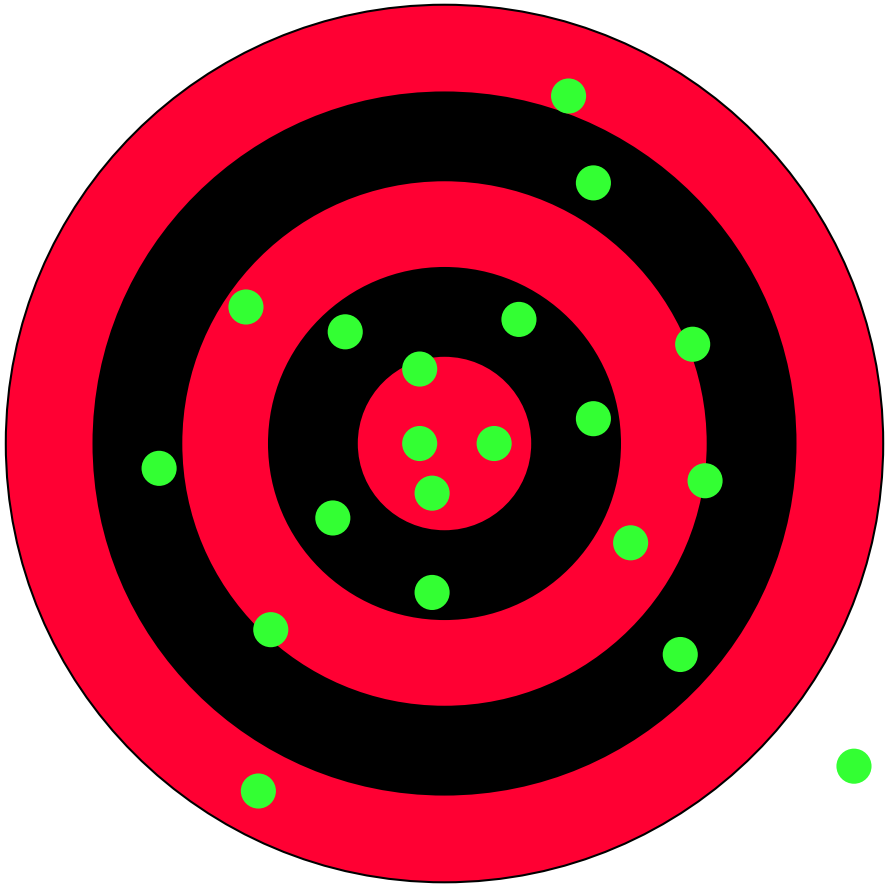


# The Target Analogy

- On any given shot, the shooter can be 95% sure that he hit the target.

# The Target Analogy

- Now suppose we are shown where the shot hit, but we are not shown where the bull's eye is.
- What is the probability that the bull's eye is within 5 inches of that shot?

# The Target Analogy

# The Target Analogy

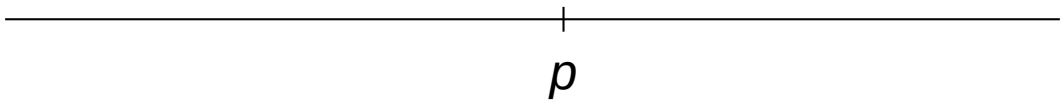
Where is the bull's eye?

# Outline

# The Confidence Interval

- In a similar way, 95% of the sample proportions $\hat{p}$ should lie within 1.96 standard deviations ($\sigma_{\hat{p}}$) of the parameter $p$.

# The Confidence Interval



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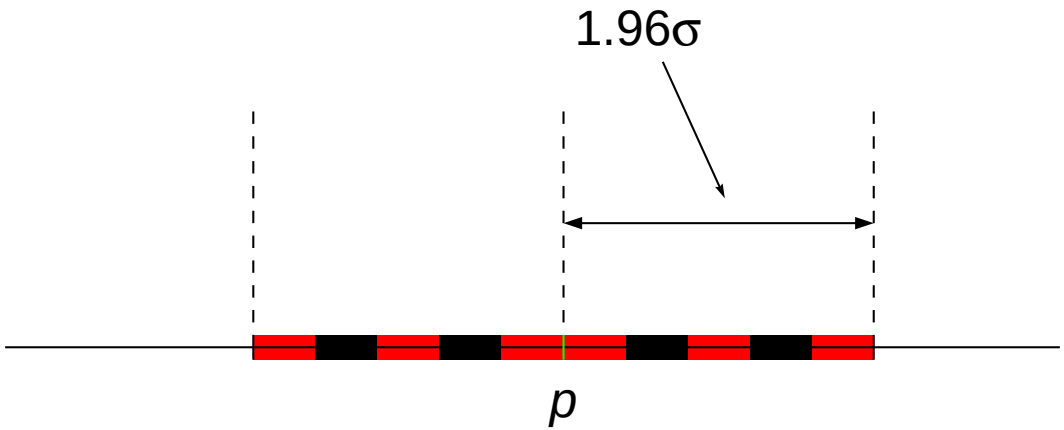

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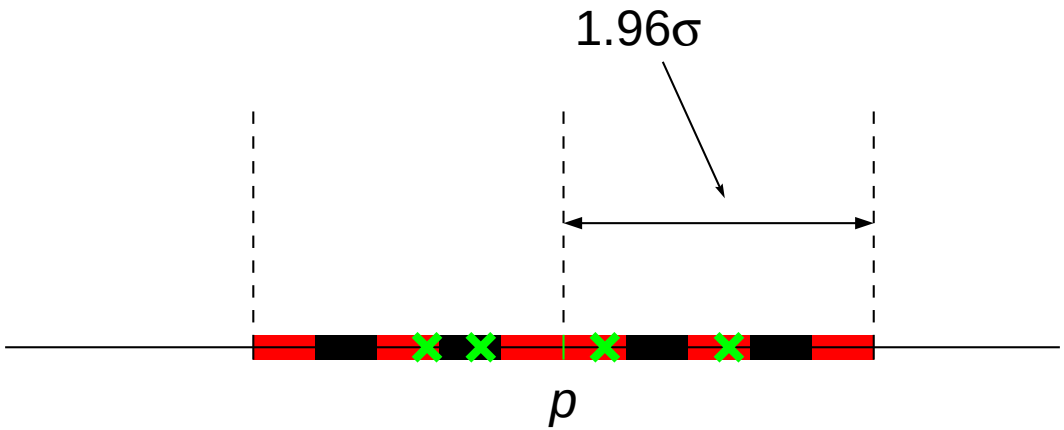

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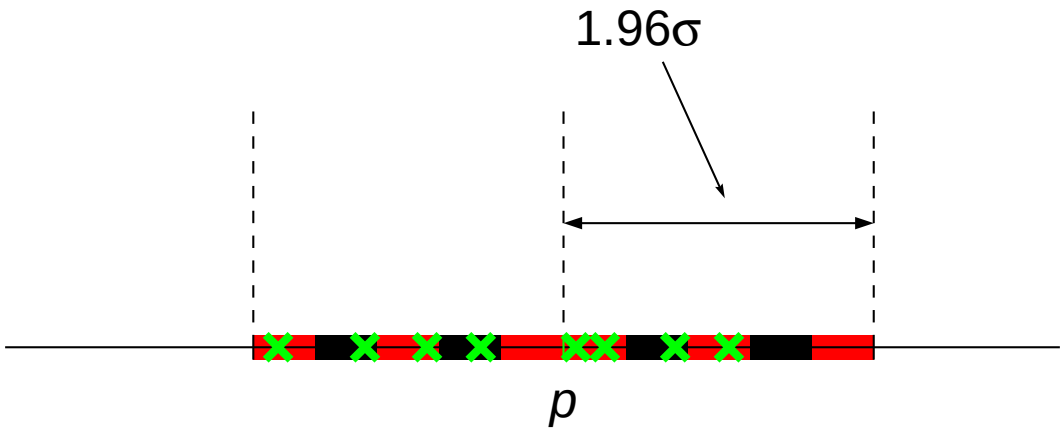

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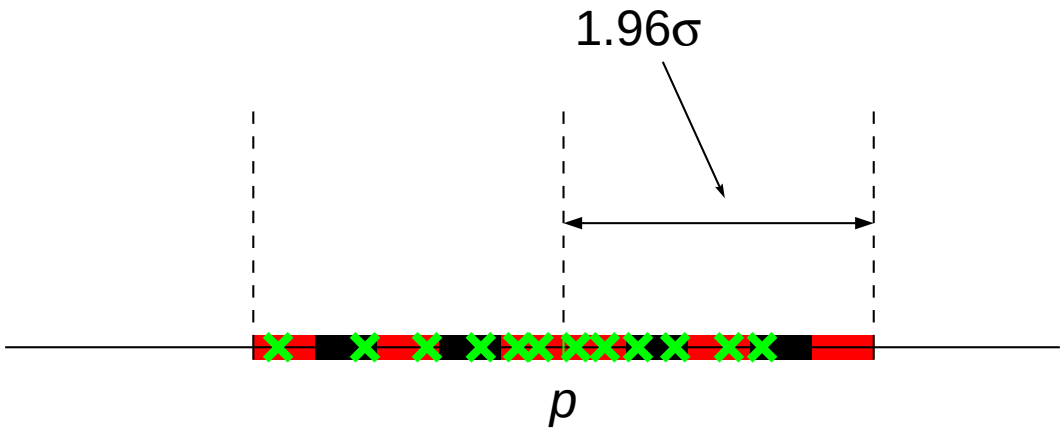

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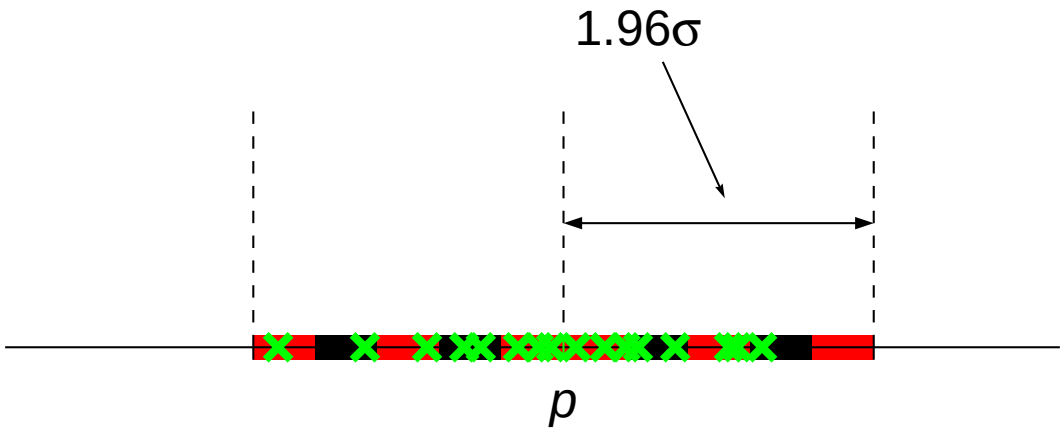

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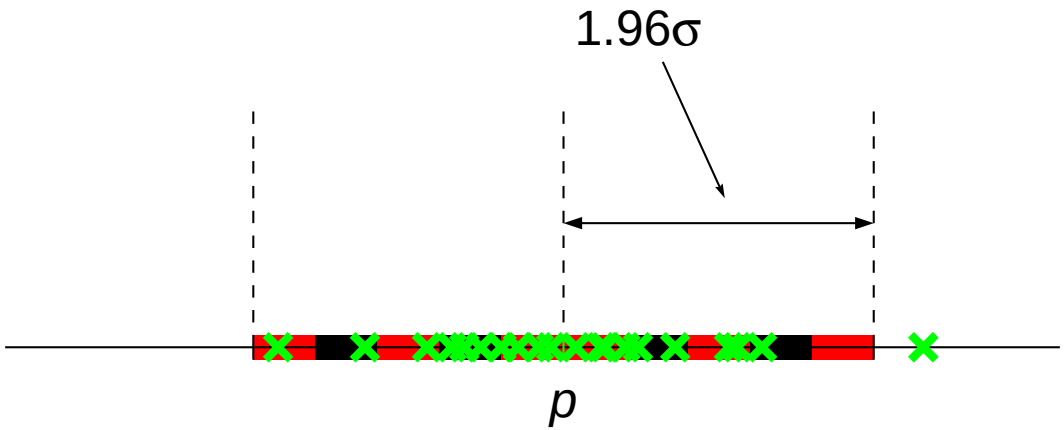

# The Confidence Interval

- Therefore, if we compute a single $\hat{p}$, then there is a 95% chance that it lies within a distance $1.96\sigma_{\hat{p}}$ of $p$.

# The Confidence Interval

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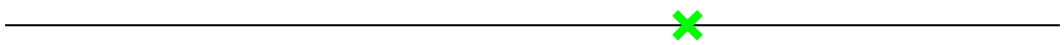

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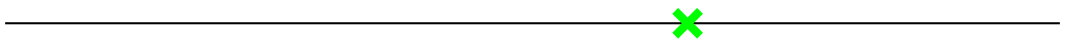
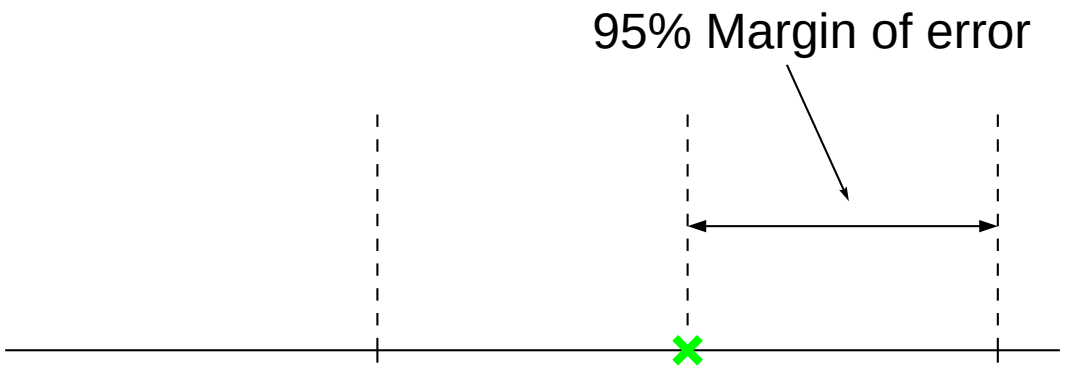


Where is $p$?

# The Confidence Interval



# Approximate 95% Confidence Intervals

- Thus, the 95% confidence interval *would* be

$$\hat{p} \pm 1.96\sigma_{\hat{p}},$$

except...

# Approximate 95% Confidence Intervals

- The formula for $\sigma_{\hat{p}}$ is

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}},$$

which requires that we know $p$, the very thing we are trying to estimate!

# Approximate 95% Confidence Intervals

- The best we can do is to use $\hat{p}$ in place of $p$ to estimate $\sigma_{\hat{p}}$:

$$\hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

# Approximate 95% Confidence Intervals

- The best we can do is to use $\hat{p}$ in place of $p$ to estimate $\sigma_{\hat{p}}$:

$$\hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

- The expression $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ is called the standard error of $\hat{p}$ and is denoted SE($\hat{p}$).

# Approximate 95% Confidence Intervals

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- The expression $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ is called the standard error of $\hat{p}$ and is denoted SE($\hat{p}$).
- It is our best approximation to $\sigma_{\hat{p}}$.

# Outline

# Example

## Example (95% confidence interval for $p$)

- Suppose we observe 525 heads out of 1000 coin tosses.
- Estimate, using a 95% confidence interval, the probability of heads.

# Example

## Example (95% confidence interval for $p$)

- The point estimate is

$$\hat{p} = \frac{525}{1000} = 0.525.$$

# Example

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- The margin of error is

$$1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 1.96\sqrt{\frac{(0.525)(0.475)}{1000}} = 0.03095.$$

# Example

## Example (95% confidence interval for $p$)

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- The margin of error is

$$1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 1.96\sqrt{\frac{(0.525)(0.475)}{1000}} = 0.03095.$$

- The confidence interval is

$$\hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.525 \pm 0.03095.$$

# Example

## Example (95% confidence interval for $p$)

- We are 95% “confident” that the point estimate of 0.525 is in error by no more than 0.03095.

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# Example

## Example (95% confidence interval for $p$)

- We are 95% “confident” that the point estimate of 0.525 is in error by no more than 0.03095.
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- Repeat the calculation for our larger sample of 5148 heads out of 10,000 tosses.

# Example

## Example (95% confidence interval for $p$)

- We are 95% “confident” that the point estimate of 0.525 is in error by no more than 0.03095.
- Note that this confidence interval includes the value 0.50 (a fair coin).
- Repeat the calculation for our larger sample of 5148 heads out of 10,000 tosses.
- Is 0.50 included?

# Outline

# Standard Confidence Levels

- The most common confidence levels are 90%, 95%, 99%, and 99.9%.

| Confidence Level | $z$ |
|---|---|
| 90% | 1.645 |
| 95% | 1.960 |
| 99% | 2.576 |
| 99.9% | 3.291 |

# The Confidence Interval

- The confidence interval is given by the formula

$$\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

  where $z$ is obtained using the `invNorm` function on the TI-83 and depends on the confidence level.
- The book denotes this value of $z$ by $z^*$.

# Probability of Error

- For a 95% confidence interval, $z_{0.025}$ represent the $2.5^{\text{th}}$ percentile (or the $97.5^{\text{th}}$ percentile) of $Z$.
- The confidence interval is

$$\hat{p} \pm z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

# Confidence Level

- Compute new confidence intervals for the 10,000 coin tosses using these confidence levels.
  - 90% confidence interval.
  - 99% confidence interval.
  - 99.9% confidence interval.

# Outline

# TI-83 - Confidence Intervals

## TI-83 Confidence intervals for $\hat{p}$

- Press `STAT`.
- Select `TESTS`.
- Select `1-PropZInt`. A display appears requesting information.
- Enter $x$, the numerator of the sample proportion.
- Enter $n$, the sample size.
- Enter the confidence level, as a decimal.
- Select `Calculate` and press `ENTER`.

# TI-83 - Confidence Intervals

## TI-83 Confidence intervals for $\hat{p}$

- A display appears with several items.
  - The title `1-PropZInt`.
  - The confidence interval, in interval notation.
  - The sample proportion $\hat{p}$.
  - The sample size.
- How would we find the margin of error?

# TI-83 - Confidence Intervals

## Example (TI-83 Confidence intervals)

- Use the TI-83 to re-do the previous examples of 10,000 coin tosses with confidence levels of 90%, 95%, 99%, and 99.9%.

# Outline

# Assignment

## Homework

- Read Section 9.4, pages 583 - 594.
- Let's Do It! 9.5.
- Exercises 19 - 25, 29, 30, page 595.
- Review exercises 39 - 44, 47 - 49, 51 - 54, page 607.